

Universal quantum processors with arbitrary radix $n \geq 2$

Alexander Yu. Vlasov

FRC/IRH, Mira Street 8, 197101, St.-Petersburg, RUSSIA

March 2001

Abstract: Here is discussed the Hamiltonian approach to construction of deterministic universal (in approximate sense) programmable quantum circuits with qubits or any other quantum systems with dimension of Hilbert space is $n \geq 2$.

Let us suggest, that universal quantum processor can be described as circuit with three “buses”: quantum, intermediate, and pseudo-classical. It is discussed with more details elsewhere (in [12] based on results about quantum control [1] and programmability [2]). Here are mentioned only topics related with theme of present paper about universality.

If we have finite set of universal (in approximate sense [3]) quantum gates, we may approximate any unitary operator with necessary precision [3, 4, 5, 6, 7, 8, 9, 10, 11]. There exist sets of two-gates both for binary quantum circuits with qubits [3, 7, 8, 9, 10] and for non-binary ones with equal dimensions $n > 2$ of Hilbert space of each elementary system (sometime it is called qunit) [11].

Here is considering application of universal set of quantum gates in three-level design of quantum processor described above. Intermediate bus together with quantum data bus is example of quantum control [1, 12], i.e., if state of intermediate bus is $|p\rangle$ then p -th gate U_p from given set is applied to quantum data bus, $|d\rangle \mapsto U_p|d\rangle$.

An approximation of given unitary operator U is decomposition on product of universal gates U_k :

$$U = U_{p_l} \cdots U_{p_2} U_{p_1} \quad (1)$$

And so it is necessary to apply quantum control l -times with value of intermediate bus is $|p_l\rangle$ on l -th step [12].

For generation of the sequence of numbers p_l may be used [12] reversible¹ classical (in terminology used by [6]) circuits acting on pseudo-classical bus together with intermediate bus. Due to this property of quantum processor it may be important sometime to have well defined algorithm of decomposition Eq. (1) for arbitrary unitary operator U . Otherwise we should keep all l numbers as table and then size of pseudo-classical bus is proportional to *maximally possible* length l_{\max} of simulation (“cyclic Q-ROM approach” [12]).

An approach with clear algorithm of decomposition uses structure of Lie algebra of unitary group [6, 8, 9, 10, 11] and from physical point of view related with Hamiltonian of quantum circuits. In such approach main object is not a unitary matrix U , but Hamiltonian of the evolution, i.e. Hermitian matrix H with property

$$U(\tau) = e^{iH\tau}, \quad \exists t, U = U(t) \quad (2)$$

for some real parameters t, τ , time.

¹It can be shown, that using irreversible classical gates on intermediate and pseudo-classical buses together with inevitable problem with quantum description of composite system with three buses may also cause non-unitary evolution of quantum data bus

The exponent Eq. (2) makes possible for small τ to use approximation with sums instead of products. Say if Hermitian matrix H can be decomposed on some basis H_I in space of Hermitian matrices, then

$$U(\tau) = \exp(i \sum_I n_I H_I \tau) \approx \prod_I \exp(i n_I H_I \tau), \quad \tau \rightarrow 0. \quad (3)$$

Let $U_I = \exp(i H_I \tau)$ and n_I may be approximated by natural number (integral part) if τ is small and n_I are big. Then:

$$\exp(i \sum_I n_I H_I \tau) \approx \prod_I U_I^{n_I}, \quad U \equiv U(t) \approx (\prod_I U_I^{n_I})^{t/\tau} \quad (4)$$

where t/τ may be considered as natural number or approximated by it for small τ .

But number of elements of basis of Hermitian matrix for quantum circuit is exponentially large. Say for system with k qubits it is 4^k and it is n^{2k} for other radix n . Due to such property, as Hamiltonians of universal quantum gates are chosen only small subset with property, that any other element of basis can be generated by using sequence of commutators

$$H_{[JK]} \equiv i(H_J H_K - H_K H_J) \quad (5)$$

To produce $U_{[JK]} = \exp(i H_{[JK]} \tau)$ it is possible to use approximation:

$$U_{[JK]} \approx U_J^{s_\tau} U_K^{s_\tau} U_J^{-s_\tau} U_K^{-s_\tau}, \quad s_\tau = 1/\sqrt{\tau} \quad (6)$$

there for small τ s_τ again can be considered as a natural number (integral part). Here small indexes like k are used for set of initial universal gates, U_1 and capital indexes like J are compound, $U_{[[12]3]}$.

Here is a problem with negative power, $-s_\tau$. It can be resolved for qubit by special design [10] where for any universal gate $U_k(t_0)$ is unit for same t_0 and so $U_k^{-s} = U_k^{t_0-s}$ with positive $t_0 - s$. For non-binary universal gates it is not necessary so, but it is possible simply to double amount of universal gates and have U_k^{-1} together with any gate U_k .

Let us describe now algorithm of approximation in general. It is chosen some small interval of time τ , the less it the higher precision. For approximation of some gate U with Hamiltonian H as in Eq. (2) may be found linear decomposition of the H by basis H_I Eq. (3). Each component with compound indexes U_I is represented by universal gates U_k as it was shown in Eq. (6), and finally approximation of U is result of nested cycles described by Eq. (3), Eq. (4).

The example shows, that approximation may demand many operations and number of steps quickly grows with refinement of precision. But if there is some method to implement structure of the approximating algorithm as reversible classical circuit, then size of program register (defined mainly by length of pseudo-classical bus) may be more appropriate. Due to it algebraic structures similar with introduced in [10, 11] may be useful.

There is also important note about reversibility of algorithm of approximation. Formally it is possible to use some irreversible circuit and apply standard technique, i.e. instead of irreversible function $f: a \mapsto f(a)$ to use reversible one with property $F: (a, 0) \mapsto (a, f(a))$, but then each step of algorithm will produce new portion of “junk” and pseudo-classical bus again must have size proportional to l_{\max} , and such a case maybe even worst, than reversible cyclic Q-ROM register without any programming.

But even if such reversible algorithm is found, the property of universality may be rather formal. It is clear from consideration above, that number of steps is proportional to amount of non-vanishing terms H_I in decomposition of Hamiltonian. So, if problem area is related with absolutely arbitrary unitary operators, then particular set of gates is not very essential and length of simulation is exponential on number of qubits.

Of course such situation would not be realistic if number of qubits is big enough and so set of basic universal gates should take into account particular set of possible problems. Say it may be task to simulate any possible quantum circuits composed by many different k -gates, with k is not very large, i.e. “local” quantum circuits.

References

- [1] A. Barenco, D. Deutsch, A. K. Ekert, and R. Jozsa, “Conditional quantum dynamics and logic gates,” *Phys. Rev. Lett.* **74**, 20, 4083–4086 (1995).
- [2] M. A. Nielsen and I. L. Chuang, “Programmable quantum gate arrays,” *Phys. Rev. Lett.* **79** 321–324, (1997).
- [3] R. Cleve, “An introduction to quantum complexity theory,” *Preprint quant-ph/9906111*, (1999).
- [4] D. Deutsch, “Quantum Theory, the Church-Turing Principle and the Universal Quantum Computer,” *Proc. R. Soc. London A* **400**, 97–117 (1985).
- [5] D. Deutsch, “Quantum Computational Networks,” *Proc. R. Soc. London A* **425**, 73–90 (1989).
- [6] D. Deutsch, A. Barenco, and A. Ekert, “Universality in quantum computation,” *Proc. R. Soc. London A* **449**, 669 (1995).
- [7] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. W. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, “Elementary gates for quantum computation,” *Phys. Rev. A* **52**, 3457–3467 (1995).
- [8] D. P. DiVincenzo, “Two-bit gates are universal for quantum computation,” *Phys. Rev. A* **51**, 1015–1022 (1995).
- [9] S. Lloyd, “Universal quantum simulators,” *Science* **273**, 1073–1078 (1996).
- [10] A. Yu. Vlasov, “Clifford algebras and universal sets of quantum gates,” E-print: [quant-ph/0010071](#) (2000) [accepted for publication in *Phys. Rev. A* (2001)].
- [11] A. Yu. Vlasov, “Noncommutative tori and universal sets of non-binary quantum gates,” E-print: [quant-ph/0012009](#) (2000).
- [12] A. Yu. Vlasov, “Classical programmability is enough for quantum circuits universality in approximate sense,” E-print: [quant-ph/0103119](#) (2001).